

Three-dimensional Flow between Two Parallel Porous Plates with Heat Transfer

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A theoretical analysis of the steady three-dimensional flow of a viscous, incompressible fluid between two parallel infinite porous horizontal plates is presented. The fluid is injected with constant velocity through the lower stationary plate and removed with a transverse sinusoidal suction velocity through the upper one in uniform horizontal motion. A series solution of the non-linear partial differential equations is obtained and discussed.

Key words: Three-dimensional; Couette Flow; Transverse; Sinusoidal; Injection/Suction.

1. Introduction

The hydrodynamic channel flow is a classical problem which has been widely discussed. The exact solution of the generalized Couette flow with a non-zero pressure gradient has been studied extensively [1]. Later, subjecting the lower stationary plate and the upper uniformly moving plate to a constant injection and suction velocity, respectively, Eckert [2] obtained an exact solution for plane Couette flow with transpiration cooling. The literature is replete with studies of two-dimensional flows between parallel plates with heat transfer. A magnetohydrodynamic flow between two parallel plates with heat transfer has been analysed by Attia and Kotb [3]. Very recently Singh [4] analysed a three-dimensional Couette flow with transpiration cooling when a transverse sinusoidal injection velocity is applied at the stationary plate of the channel. In the present work, the steady flow of viscous incompressible fluid between two parallel infinite horizontal plates has been studied in the presence of a pressure gradient. The lower porous plate is subjected to a constant injection velocity and the upper porous plate in uniform motion to a transverse sinusoidal suction velocity.

2. Mathematical Analysis

We consider a generalized Couette flow of a viscous incompressible fluid between two parallel horizontal porous plates with non-zero pressure gradient as shown

in Fig. 1 of [4]. Denoting the velocity components in the x , y , z directions by u , v , w , respectively, and the temperature by θ , the problem is governed by the non-dimensional equations

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (1)$$

$$v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = G + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad (2)$$

$$v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}, \quad (3)$$

$$v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}, \quad (4)$$

$$v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = \frac{1}{\text{Pr}} \left(\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right), \quad (5)$$

where

$$y = \frac{y^*}{h}, \quad z = \frac{z^*}{h}, \quad u = \frac{u^* h}{v}, \quad v = \frac{v^* h}{v}, \quad w = \frac{w^* h}{v}, \\ \theta = \frac{T^* - T_0^*}{T_1^* - T_0^*}, \quad p = \frac{p^* h^2}{\rho v^2}, \quad G = -h \frac{\partial p}{\partial x^*},$$

$$\text{Pr (Prandtl number)} = \frac{\mu C_p}{k}.$$

The boundary conditions to the problem in dimensionless form are

$$\left. \begin{aligned} y=0; & \quad u=0, \quad v=\lambda, \quad w=0, \quad \theta=0, \\ y=1; & \quad u=\text{Re}, \quad v=\lambda(1+\epsilon \cos \pi z), \quad w=0, \quad \theta=1. \end{aligned} \right\}, \quad (6)$$

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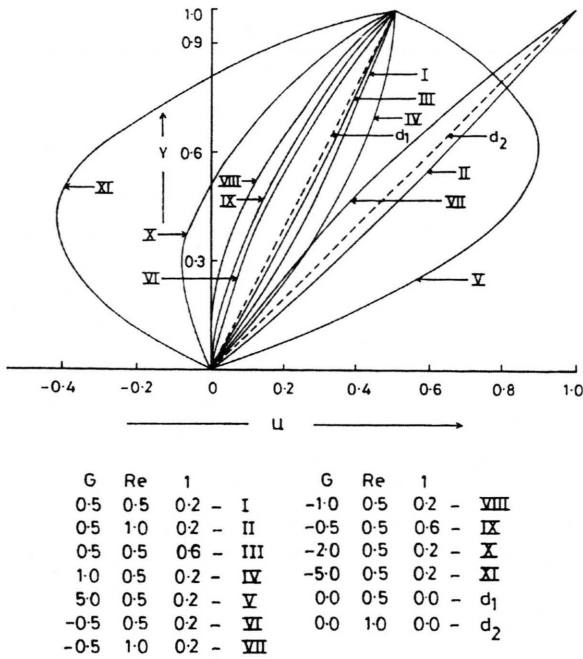


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Fig. 1. Main flow velocity profiles for $z = 0$ and $\varepsilon = 0.2$.

where Re (Reynolds number) = $\frac{Uh}{\nu}$, λ (injection/suction parameter) = $\frac{Vh}{\nu}$.

Since the amplitude ε ($\ll 1$) of the injection/suction velocity at the stationary plate is very small, we now assume a solution of the form

$$f(y, z) = f_0(y) + \varepsilon f_1(y, z) + \varepsilon^2 f_2(y, z) + \dots, \quad (7)$$

where f stands for any of u, v, w, p and θ . When $\varepsilon = 0$, the problem reduces to the two-dimensional flow with constant injection and suction at both the plates. The solution of this two-dimensional problem is

$$u_0(y) = \frac{\lambda Re - G}{\lambda(e^\lambda - 1)}(e^{\lambda y} - 1) + \frac{Gy}{\lambda}, \quad v_0 = \lambda, \quad (8)$$

$w_0 = 0$, and $p_0 = \text{constant}$,

$$\theta_0(y) = \frac{e^{\lambda Pr y} - 1}{e^{\lambda Pr} - 1}. \quad (9)$$

When $\varepsilon \neq 0$, substituting (7) in (1) to (5) and comparing the coefficients of identical powers of ε , neglecting those of $\varepsilon^2, \varepsilon^3$ etc., the following first order equations are obtained with the help of (8) and (9):

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \quad (10)$$

$$v_1 \frac{\partial u_0}{\partial y} + \lambda \frac{\partial u_1}{\partial y} = \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}, \quad (11)$$

$$\lambda \frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}, \quad (12)$$

$$\lambda \frac{\partial w_1}{\partial y} = -\frac{\partial p_1}{\partial z} + \frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2}, \quad (13)$$

$$v_1 \frac{\partial \theta_0}{\partial y} + \lambda \frac{\partial \theta_1}{\partial y} = \frac{1}{Pr} \left(\frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2} \right). \quad (14)$$

The corresponding boundary conditions reduce to

$$\left. \begin{aligned} y = 0; & \quad u_1 = 0, \quad v_1 = 0, \quad w_1 = 0, \quad \theta_1 = 0, \\ y = 1; & \quad u_1 = 0, \quad v_1 = \lambda \cos \pi z, \quad w_1 = 0, \quad \theta_1 = 0. \end{aligned} \right\} \quad (15)$$

These are the linear partial differential equations describing the three-dimensional cross flow.

In order to solve these equations we shall first consider (10), (12) and (13) for cross flow, being independent of the main flow component u_1 and the temperature field θ_1 . We assume v_1, w_1 and p_1 of the form

$$v_1(y, z) = v_{11}(y) \cos \pi z, \quad (16)$$

$$w_1(y, z) = -\frac{1}{\pi} v'_{11}(y) \sin \pi z, \quad (17)$$

$$p_1(y, z) = p_{11}(y) \cos \pi z, \quad (18)$$

where a prime denotes differentiation with respect to y . Equations (16) and (17) for $v_1(y, z)$ and $w_1(y, z)$, respectively, have been chosen so that the equation of continuity is satisfied. Substituting these equations into (12) and (13) and applying the corresponding transformed boundary conditions, we get the solutions of v_1, w_1 and p_1 as

$$v_1(y, z) = \left(C e^{\eta_1 y} + D e^{\eta_2 y} - \frac{A}{\lambda} e^{\pi y} - \frac{B}{\lambda} e^{-\pi y} \right) \cos \pi z, \quad (19)$$

$$w_1(y, z) = -\frac{1}{\pi} \left(C \eta_1 e^{\eta_1 y} + D \eta_2 e^{\eta_2 y} - \frac{A\pi}{\lambda} e^{\pi y} + \frac{B\pi}{\lambda} e^{-\pi y} \right) \sin \pi z, \quad (20)$$

$$p_1(y, z) = (A e^{\pi y} + B e^{-\pi y}) \cos \pi z, \quad (21)$$

where

$$A = \frac{\pi}{A^*} \left[e^{-\pi} (r_1 - r_2) + \pi (e^{\eta_1} - e^{\eta_2}) + r_2 e^{\eta_2} - r_1 e^{\eta_1} \right],$$

$$B = \frac{\pi}{A^*} \left[e^{\pi} (r_1 - r_2) - \pi (e^{\eta_1} - e^{\eta_2}) + r_2 e^{\eta_2} - r_1 e^{\eta_1} \right],$$

$$C = \frac{\pi}{\lambda A^*} \left[\pi (e^{-\pi} - e^{\pi}) + r_2 (2e^{r_2} - e^{\pi} - e^{-\pi}) \right],$$

$$D = -\frac{\pi}{\lambda A^*} \left[\pi (e^{-\pi} - e^{\pi}) + r_1 (2e^{r_1} - e^{\pi} - e^{-\pi}) \right],$$

$$A^* = \frac{\pi}{\lambda^2} \left[(r_2 - r_1) \cdot \left\{ 2 + 2e^{r_1+r_2} - e^{\pi+r_2} - e^{r_2-\pi} - e^{\pi+r_1} - e^{r_1-\pi} \right\} + 2\pi \left\{ e^{\pi+r_2} - e^{r_2-\pi} - e^{\pi+r_1} - e^{r_1-\pi} \right\} \right],$$

$$r_1 = \frac{1}{2} \left(\lambda + \sqrt{\lambda^2 + 4\pi^2} \right), \quad r_2 = \frac{1}{2} \left(\lambda - \sqrt{\lambda^2 + 4\pi^2} \right).$$

Now for the main flow velocity component $u_1(y, z)$ and the temperature field $\theta_1(y, z)$, respectively, we assume

$$u_1(y, z) = u_{11}(y) \cos \pi z, \quad (22)$$

$$\theta_1(y, z) = \theta_{11}(y) \cos \pi z. \quad (23)$$

Substituting (22) and (23) in (11) and (14), we obtain the solutions under the corresponding transformed boundary conditions as

$$u_1(y, z) = \left[M_1 e^{r_1 y} + M_2 e^{r_2 y} + \frac{(\lambda \text{Re} - G)}{(e^{\lambda} - 1)} \cdot \left\{ \frac{C e^{(\lambda+r_1)y}}{2\lambda r_1} + \frac{D e^{(\lambda+r_2)y}}{2\lambda r_2} - \frac{A e^{(\lambda+\pi)y}}{\pi \lambda^2} + \frac{B e^{(\lambda-\pi)y}}{\pi \lambda^2} \right\} + \frac{G}{\lambda} \left\{ \frac{C y e^{r_1 y}}{2r_1 - \lambda} + \frac{D y e^{r_2 y}}{2r_2 - \lambda} + \frac{A e^{\pi y}}{\lambda^2 \pi} - \frac{B e^{-\pi y}}{\lambda^2 \pi} \right\} \right] \cos \pi z, \quad (24)$$

$$\theta_1(y, z) = \left[N_1 e^{s_1 y} + N_2 e^{s_2 y} + \frac{\text{Pr}^2}{(e^{\lambda \text{Pr}} - 1)} \cdot \left\{ \frac{C \lambda e^{(\lambda \text{Pr} + r_1)y}}{2\lambda \text{Pr} r_1} + \frac{D \lambda e^{(\lambda \text{Pr} + r_2)y}}{2\lambda \text{Pr} r_2} - \frac{A e^{(\lambda \text{Pr} + \pi)y}}{\pi \lambda \text{Pr}} + \frac{B e^{(\lambda \text{Pr} - \pi)y}}{\pi \lambda \text{Pr}} \right\} \right] \cos \pi z, \quad (25)$$

where

$$s_1 = \frac{1}{2} \left(\lambda \text{Pr} + \sqrt{\lambda^2 \text{Pr}^2 + 4\pi^2} \right),$$

$$s_2 = \frac{1}{2} \left(\lambda \text{Pr} - \sqrt{\lambda^2 \text{Pr}^2 + 4\pi^2} \right),$$

$$M_1 = -\frac{(\lambda \text{Re} - G)}{(e^{\lambda} - 1)(e^{r_1} - e^{r_2})} \left[\frac{C(e^{\lambda+r_1} - e^{r_2})}{r_1^2 + \lambda r_1 - \pi^2} + \frac{D(e^{\lambda+r_2} - e^{r_1})}{r_2^2 + \lambda r_2 - \pi^2} - \frac{A(e^{\lambda+\pi} - e^{r_2})}{\pi \lambda^2} + \frac{B(e^{\lambda-\pi} - e^{r_1})}{\pi \lambda^2} \right] - \frac{G}{\lambda(e^{r_1} - e^{r_2})} \cdot \left[\frac{C e^{r_1}}{2r_1 - \lambda} + \frac{D e^{r_2}}{2r_2 - \lambda} + \frac{A(e^{\pi} - e^{r_2})}{\pi \lambda^2} - \frac{B(e^{-\pi} - e^{r_1})}{\pi \lambda^2} \right],$$

$$M_2 = \frac{(\lambda \text{Re} - G)}{(e^{\lambda} - 1)(e^{r_1} - e^{r_2})} \left[\frac{C(e^{\lambda+r_1} - e^{r_1})}{r_1^2 + \lambda r_1 - \pi^2} + \frac{D(e^{\lambda+r_2} - e^{r_1})}{r_2^2 + \lambda r_2 - \pi^2} - \frac{A(e^{\lambda+\pi} - e^{r_1})}{\pi \lambda^2} + \frac{B(e^{\lambda-\pi} - e^{r_1})}{\pi \lambda^2} \right] + \frac{G}{\lambda(e^{r_1} - e^{r_2})} \cdot \left[\frac{C e^{r_1}}{2r_1 - \lambda} + \frac{D e^{r_2}}{2r_2 - \lambda} + \frac{A(e^{\pi} - e^{r_1})}{\pi \lambda^2} - \frac{B(e^{-\pi} - e^{r_1})}{\pi \lambda^2} \right],$$

$$N_1 = \frac{-\text{Pr}^2}{(e^{\lambda \text{Pr}} - 1)(e^{s_2} - e^{s_1})} \cdot \left[\frac{C \lambda (e^{s_2} - e^{\lambda \text{Pr} + r_1})}{r_1^2 + \lambda \text{Pr} r_1 - \pi^2} + \frac{D \lambda (e^{s_2} - e^{\lambda \text{Pr} + r_2})}{r_2^2 + \lambda \text{Pr} r_2 - \pi^2} - \frac{A(e^{s_2} - e^{\lambda \text{Pr} + \pi})}{\pi \lambda \text{Pr}} + \frac{B(e^{s_2} - e^{\lambda \text{Pr} - \pi})}{\pi \lambda \text{Pr}} \right],$$

$$N_2 = \frac{\text{Pr}^2}{(e^{\lambda \text{Pr}} - 1)(e^{s_2} - e^{s_1})} \cdot \left[\frac{C \lambda (e^{s_1} - e^{\lambda \text{Pr} + r_1})}{r_1^2 + \lambda \text{Pr} r_1 - \pi^2} + \frac{D \lambda (e^{s_1} - e^{\lambda \text{Pr} + r_2})}{r_2^2 + \lambda \text{Pr} r_2 - \pi^2} - \frac{A(e^{s_1} - e^{\lambda \text{Pr} + \pi})}{\pi \lambda \text{Pr}} + \frac{B(e^{s_1} - e^{\lambda \text{Pr} - \pi})}{\pi \lambda \text{Pr}} \right].$$

Now, after knowing the velocity field we can calculate the skin-friction components τ_x and τ_z in the main flow and transverse directions, respectively, as

$$\begin{aligned} \tau_x &= \frac{h \cdot \tau_x^*}{\mu U} = \left(\frac{du_0}{dy} \right)_{y=0} + \varepsilon \left(\frac{du_{11}}{dy} \right)_{y=0} \cos \pi z \\ &= \frac{\lambda \text{Re} - G}{e^{\lambda} - 1} + \frac{G}{\lambda} + \varepsilon \left[M_1 r_1 + M_2 r_2 + \frac{(\lambda \text{Re} - G)}{(e^{\lambda} - 1)} \cdot \left\{ \frac{C(\lambda + r_1)}{r_1^2 + \lambda r_1 - \pi^2} + \frac{D(\lambda + r_2)}{r_2^2 + \lambda r_2 - \pi^2} - \frac{A(\lambda + \pi)}{\pi \lambda^2} + \frac{B(\lambda - \pi)}{\pi \lambda^2} \right\} + \frac{G}{\lambda} \left\{ \frac{C}{2r_1 - \lambda} + \frac{D}{2r_2 - \lambda} + \frac{A}{\lambda^2} + \frac{B}{\lambda^2} \right\} \right] \cos \pi z, \quad (26) \end{aligned}$$

$$\begin{aligned} \tau_z &= \frac{h \cdot \tau_z^*}{\mu V} = \varepsilon \left(\frac{\partial w_1}{\partial y} \right)_{y=0} \\ &= -\frac{\varepsilon}{\pi} \left(C r_1^2 + D r_2^2 - \frac{A \pi^2}{\lambda} - \frac{B \pi^2}{\lambda} \right) \sin \pi z. \quad (27) \end{aligned}$$

From the temperature field we can calculate the rate of heat transfer in terms of the Nusselt number as

$$\text{Nu} = \frac{h \cdot q_w^*}{k(T_1^* - T_0^*)} = \left(\frac{d\theta_0}{dy} \right)_{y=0} + \varepsilon \left(\frac{d\theta_{11}}{dy} \right)_{y=0} \cos \pi z,$$

$$Nu = \frac{\lambda Pr}{e^{\lambda Pr} - 1} + \varepsilon \left[N_1 s_1 + N_2 s_2 + \frac{Pr^2}{e^{\lambda Pr} - 1} \right. \\ \left. \cdot \left\{ \frac{C\lambda(\lambda Pr + r_1)}{r_1^2 + \lambda Pr r_1 - \pi^2} + \frac{D\lambda(\lambda Pr + r_2)}{r_2^2 + \lambda Pr r_2 - \pi^2} \right. \right. \\ \left. \left. - \frac{A(\lambda Pr + \pi)}{\pi \lambda Pr} + \frac{B(\lambda Pr - \pi)}{\pi \lambda Pr} \right\} \right] \cos \pi z. \quad (28)$$

3. Discussion of Predictions

The main flow velocity distribution u against the vertical distance y for $G \geq 0$ is shown in Figure 1. As expected in the absence of the pressure gradient and the injection/suction velocity at the plates, i.e. for $G = 0$ and $\lambda = 0$, the velocity distribution becomes linear. The lines d_1 and d_2 represent the case of simple Couette flow for a dimensionless upper plate velocity corresponding to $Re = 0.5$ and $Re = 1.0$, respectively. For $G > 0$, i.e. for a pressure decreasing in the direction of the flow or, in other words, for a favourable pressure gradient, the velocity distribution is positive over the whole width of the channel, as can be seen in Figure 1. For $G < 0$, i.e. for a pressure increasing in the direction of the flow, or in other words for an adverse pressure gradient, we find that a back flow begins to occur near the stationary plate as $G < -1$. We observe from this figure that the velocity increases with increase of the Reynolds number Re irrespective of the favourable or adverse pressure gradient [curves (I, II) and (VI, VII)]. Keeping the velocity of the upper plate $Re = 0.5$ fixed, the velocity is observed to be decreasing with the increase of the injection/suction velocity parameter λ both for $G > 0$ or $G < 0$ [curves (I, III) and (VI, IX)]. It is evident from this figure that for $G = -5$, a reversed flow is marked for a circulation depth equal to 0.41.

The cross flow velocity is presented in Fig. 2 for $z = -0.5$. It is found that with increase of the injection/suction velocity parameter λ , the cross flow velocity increases.

The main flow and transverse components of the skin friction namely, τ_x and τ_z respectively, and Nusselt number (Nu) are plotted in Figure 3. We observe that τ_x increases with the increase of G (curves I, II) as well as with the increase of Re (curves I, III). The transverse component of skin friction is observed to be $-ve$ for $z = 0.5$ (curve IV). From the curves V and VI it is clear that the Nusselt number is quite less in water ($Pr = 7.0$)

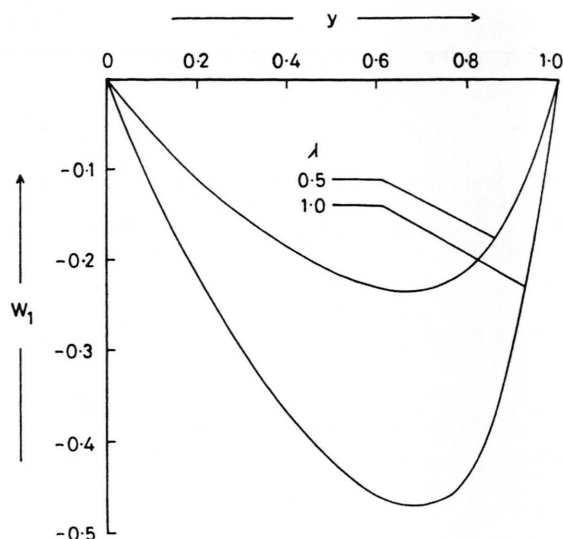


Fig. 2. Cross flow velocity for $z = 0.5$ and $\varepsilon = 0.2$.

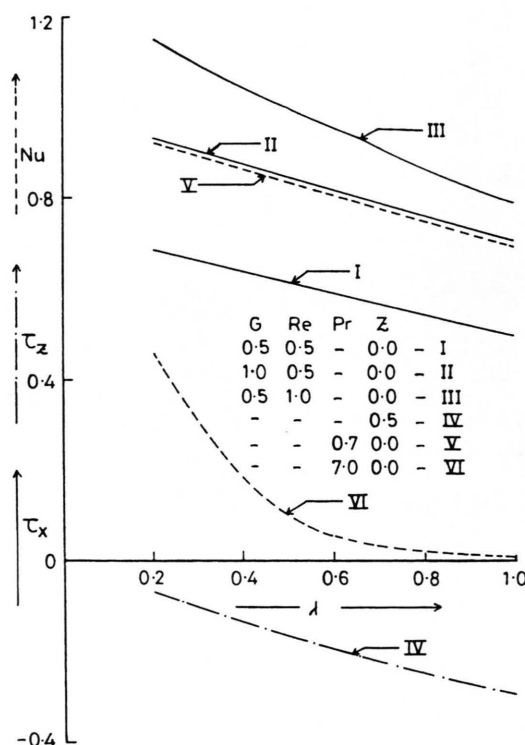


Fig. 3. Main flow and transverse components of skin friction τ_x (—), τ_z (-.-.-) and Nusselt number Nu (-----) for $\varepsilon = 0.2$.

than in air ($Pr = 0.7$). We also observe that τ_x , τ_z and the Nusselt number decrease with increase of the injection/suction velocity parameter λ .

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